

CHIRAL SYMMETRY AND SPECTRUM OF EUCLIDEAN DIRAC OPERATOR

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After recalling some connections between the Spontaneous Breakdown of Chiral Symmetry (SB χ S) and the spectrum of the Dirac operator for Euclidean QCD on a torus, we use this tool to reconsider two related issues : the Zweig rule violation in the scalar channel and the dependence of SB χ S order parameters on the number N_f of massless flavours¹. The latter would result into a great variety of SB χ S patterns in the (N_f, N_c) plane, which could be studied through Leutwyler-Smilga sum rules in association with lattice computations of the Dirac spectrum².

1 Chiral symmetry and its breakdown

At low energies, QCD cannot be described in a perturbative way : its degrees of freedom (quarks and gluons) are different from its asymptotic states (hadrons). To analyse the theory, one can fortunately follow another path and consider its symmetries. If we set to zero the masses of the N_f lightest quarks (chiral limit), the right- and left-handed fermions can be rotated separately in flavour space and QCD exhibits a chiral symmetry $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$. This symmetry is spontaneously broken along its axial directions by the fundamental state of the theory, generating pseudoscalar Goldstone bosons. When light quark masses are turned on, they appear in the QCD Lagrangian as small perturbations breaking down explicitly chiral symmetry and providing the Goldstone bosons a light mass.

In Nature, $N_f = 3$ flavours are light enough in comparison with QCD scale (~ 1 GeV) for this treatment, and the Goldstone bosons are identified with the light pseudoscalar octet (π, K, η) . It is often stated that SB χ S is triggered by a large condensation of quark-antiquark pairs in the vacuum: $\langle 0 | \bar{q}q | 0 \rangle = -\Sigma(N_f)$, but this assumption has to (and is about to) be tested experimentally, at least in the two-flavour case³. Supposing the leadership of the quark condensate in SB χ S, the Standard Chiral Perturbation Theory (S χ PT)⁴ provides an effective description of the pseudoscalar-octet phenomenology. With the same goal, the Generalized χ PT¹¹ keeps the quark condensate as a free (and possibly small) parameter and, at a fixed order of accuracy, takes into account contributions neglected by S χ PT.

We highlight the particular status of the two-point correlator

$$\Xi_{\mu\nu}(q^2)\delta^{ij} = i \int d^4x e^{iq\cdot x} \langle \Omega | T \{ V_\mu^i(x) V_\nu^j(0) - A_\mu^i(x) A_\nu^j(0) \} | \Omega \rangle \quad (1)$$

where V (A) are the Noether currents for vector (axial) rotations in flavour space, and $|\Omega\rangle$ is the vacuum of the massive theory. $\Xi_{\mu\nu}$ is a chiral order parameter for any momentum q . In the chiral limit, a massless pole arises because of the Goldstone bosons coupling to the axial current. $\Xi_{\mu\nu}(0)$ tends then to $-\eta_{\mu\nu} F^2(N_f)$, yielding the decay constant of the Goldstone bosons for N_f massless flavours. A non-vanishing $\Xi_{\mu\nu}(0)$ is thus a necessary and sufficient condition for SB χ S, by contrast with other order parameters, whose vanishing is not a clear signal for the full symmetry restoration.

2 N_f -dependence of SB χ S order parameters

A common approximation leads to neglect the (supposed) weak N_f -dependence of SB χ S order parameters due to light quark loops: this effect is suppressed in the large- N_c limit and violates the Zweig rule, both considered as good approximations of QCD⁵. On the other hand, the scalar channel does not obey large- N_c predictions, and all chiral order parameters should vanish for N_f/N_c large enough, where the perturbative QCD β -function predicts the end of confinement. Furthermore, recent lattice simulations⁶ find large variations in the SB χ S pattern when N_f increases, and a sum rule estimate⁷, based on experimental data in the scalar channel, asserts $\Sigma(N_f)$ could decrease by half from 2 to 3 flavours.

To study this problem, Euclidean QCD on a torus (of volume $V = L^4$) seems an appealing tool, because SB χ S is related to the lower end of the spectrum of the Dirac operator: $H[G] = \gamma_\mu(\partial_\mu + iG_\mu)$. For each gauge configuration, this Hermitean operator admits only real eigenvalues (e.v.s) λ_n . The spectrum is symmetrical with respect to zero, because $\{H, \gamma_5\} = 0$, and the winding number $\nu[G]$ of the gauge configuration provides the number of exactly zero e.v.s. We label positive e.v.s in the ascending order with a positive integer, and denote the negative part of the spectrum $\lambda_{-n} = -\lambda_n$.

In a d -dimensional space, Dirac e.v.s turn out to be uniformly bound by the free theory⁹: $|\lambda_n[G]| < \omega_n \equiv C \cdot n^{1/d}/L$, where C depends only on the geometry of the space-time manifold, but neither on G , n nor $V = L^d$. This bound shows the paramagnetic response of the Dirac spectrum to an external gauge field. We see the lowest e.v.s accumulate around zero for $L \rightarrow \infty$.

The eigenvalues λ and the eigenvectors $\phi(x)$ of the Dirac operator are independent of quark flavours and quark masses, and they can be used to

project and integrate fermionic fields in any functional integral. The following propagators and the (normalized) determinants are involved:

$$S_j(x, y|G) = \sum_n \frac{\phi_n(x)\phi_n^\dagger(y)}{m_j - i\lambda_n[G]}, \quad \Delta(m|G) = m^{|\nu|} \prod_{n>0} \frac{m^2 + \lambda_n^2[G]}{m^2 + \omega_n^2}, \quad (2)$$

In particular, the fermionic determinant reads: $\det(M - iH) = \prod_j \Delta(m_j|G)$. In the chiral limit $m_1 = \dots = m_{N_f} = m \rightarrow 0$ (the other quark masses remaining fixed), $\langle \bar{q}q \rangle$ yields⁸:

$$\Sigma(N_f) = \underline{\lim} \frac{1}{L^4} \ll \int dx \text{Tr } S_1(x, x|G) \gg_{N_f} \underline{\lim} \frac{1}{L^4} \ll \sum_n \frac{m}{m^2 + \lambda_n^2} \gg_{N_f} \quad (3)$$

where $\underline{\lim}$ means the limit $L \rightarrow \infty$, followed by $m \rightarrow 0$. The average over gauge configurations is:

$$\ll \Gamma \gg_{N_f} = Z^{-1} \int d\mu[G] e^{-S[G]} \Gamma \Delta^{N_f}(m|G) \prod_{j>N_f} \Delta(m_j|G), \quad (4)$$

where the dependence on N_f is explicit. This average is conveniently normalized ($\ll 1 \gg_{N_f} = 1$) and requires a non-perturbative regularization and renormalization. In a similar way, (1) once projected yields¹⁰

$$F^2(N_f) = \underline{\lim} \frac{1}{L^4} \ll \sum_{k,n} \frac{m}{m^2 + \lambda_k^2} \frac{m}{m^2 + \lambda_n^2} J_{kn} \gg_{N_f}, \quad (5)$$

with the transition probability: $J_{kn} = \frac{1}{4} \sum_\mu |\int dx \phi_k^\dagger(x) \gamma_\mu \phi_n(x)|^2$. In the chiral limit $m \rightarrow 0$, F^2 and Σ are essentially dominated by the lowest Dirac e.v.s, because of the factors $m/(m^2 + \lambda^2)$. The quark condensate is only sensitive to the e.v.s accumulating like $1/L^4$ in the gluonic average, while $1/L^2$ is a sufficient speed for an e.v. to contribute to F^2 .

In $\Delta(m|G)$, infrared and ultraviolet e.v.s can be split with respect to a cutoff Λ . If K is the integer such as $\omega_K = \Lambda$, we have

$$\Delta = m^{|\nu|} \Delta_{\text{IR}}(m|G) \Delta_{\text{UV}}(m|G), \quad \Delta_{\text{IR}}(m|G) = \prod_{k=1}^K \frac{m^2 + \lambda_k^2[G]}{m^2 + \omega_k^2} < 1. \quad (6)$$

Δ_{UV} needs regularization and renormalization. However, in (4), order parameters dominated by the lowest Dirac e.v.s should be essentially sensitive to Δ_{IR} , which is bounded by 1 and more and more efficiently suppressed when N_f goes up. Infrared-dominated order parameters should then decrease: $\Sigma(N_f + 1) < \Sigma(N_f)$ and $F^2(N_f + 1) < F^2(N_f)$. Moreover, since the quark

condensate is more sensitive to the lowest e.v.s than F^2 , its suppression should be stronger.

The dependence of $\Sigma(N_f)$ on the $(N_f + 1)$ -th quark, called s , is given by the connected part of the correlator:

$$\frac{\partial}{\partial m_s} \Sigma(N_f) = \lim_{m \rightarrow 0} \int dx \langle 0 | \bar{u}u(x) \bar{s}s(0) | 0 \rangle^c \equiv \Pi_Z(m_s) \quad (7)$$

so that $\Sigma(N_f) = \Sigma(N_f + 1) + \int_0^{m_s} d\mu \Pi_Z(\mu)$. If the number of flavours lies just below $n_{\text{crit}}(N_c)$, the large difference $\Sigma(N_f) - \Sigma(N_f + 1)$ implies, through Π_Z , a significant Zweig rule violation in the 0^{++} channel which is actually observed for $N_f = 2 - 3$. Besides, the large- N_c limit throws n_{crit} to infinity: if $n_{\text{crit}}(3)$ lies near 3, the $1/N_c$ expansion should converge very slowly to the phenomenology in the scalar sector. The study of chiral phase transitions, far from being academic, could thus deepen our understanding of actual QCD.

3 Leutwyler-Smilga sum rules : extensions and applications

In the (N_f, N_c) plane, $\text{SB}\chi\text{S}$ order parameters could therefore exhibit very different behaviours for their decrease and their vanishing, leading to a rich chiral phase structure for QCD. The lowest Dirac eigenvalues may shed some light on this problem, through sum rules first described by Leutwyler and Smilga¹². In a large torus, the lightest excitations (the pseudoscalar mesons) dominate the partition function of Euclidean QCD, because heavier states are exponentially suppressed. Moreover, since periodic boundary conditions are chosen, the effective Lagrangian density \mathcal{L}_{eff} describing the Goldstone bosons in a box with periodic boundary conditions is identical to its equivalent in an infinite volume¹³ : neither Lorentz-breaking terms, nor volume-dependent coefficients arise, once on the torus.

The QCD and χPT representations of the partition function can thus be matched for large volumes

$$\int d\mu[G] e^{-S[G]} \det(-i\not{D} + \tilde{M}) \sim \int d\mu[U] \exp \left[-S_{\text{eff}}(U, \partial U, M e^{i\theta/N_f}) \right], \quad (8)$$

where $U(x) \in \text{SU}(N_f)$ collects the pseudo-Goldstone bosons, $\tilde{M} = (1 - \gamma_5)M/2 + (1 + \gamma_5)M^\dagger/2$ is the light quark mass matrix (with positive real eigenvalues), and θ the vacuum angle, conjugate to the winding number $\nu[G]$ in the QCD Lagrangian. (8) leads to sum rules for $\langle \sigma_k \rangle_\nu$:

$$\sigma_k = \sum_{n>0} \frac{1}{\lambda_n^k}, \quad \langle \Gamma \rangle_\nu = \mathcal{A}_\nu \int_\nu d\mu[G] e^{-S[G]} \left(\prod_{n>0} \lambda_n^2 \right)^{N_f} \Gamma, \quad (9)$$

where the functional integral is performed over the gluonic configurations with a fixed winding number ν , and \mathcal{A}_ν is a normalization constant such that $\langle 1 \rangle_\nu = 1$.

The sums $\langle \sigma_k \rangle_\nu$ are especially sensitive to the lowest Dirac eigenvalues, and should therefore reflect SB χ S. Actually, Leutwyler and Smilga obtained the asymptotic behaviour of such sums for $L \rightarrow \infty$, for instance¹²

$$\langle \sigma_2 \rangle_\nu \rightarrow \frac{(V\Sigma)^2}{4[N_f + |\nu|]}, \quad \langle \sigma_4 \rangle_\nu \rightarrow \frac{(V\Sigma)^4}{16[N_f + |\nu|][(N_f + |\nu|)^2 - 1]}, \quad (10)$$

Two questions arise: how would these limits change in a phase where chiral symmetry is broken but $\langle \bar{q}q \rangle$ vanishes? what are the finite-size corrections to these asymptotic results?

(10) were derived by considering only the leading order of S_{eff} in S χ PT. The finite-size corrections to the sum rules are due to higher orders and should be hard to disentangle even at intermediate volumes, if their contributions are small compared to the one due to the quark condensate. On the other hand, a small $\langle \bar{q}q \rangle$ leave space for large finite-size effects. G χ PT is designed for this situation¹¹ and *e.g.* for σ_4 at $\nu = 0$, one ends up with the sum rule²:

$$\begin{aligned} \langle \sigma_4 \rangle_0 = \frac{V^2}{16N_f(N_f^2 - 1)} & \left\{ (V\Sigma^2)^2 + (V\Sigma^2) \cdot 4F^2[3Z_S - Z_P - N_f A] \right. \\ & \left. + 4F^4[3(Z_S^2 + Z_P^2) - 2Z_S Z_P + A^2 - 2N_f(Z_S + Z_P)A] \right\} + \dots \end{aligned} \quad (11)$$

where A , Z_S and Z_P are chiral order parameters arising in the effective Lagrangian. The dots remind of higher order terms: (11) neglects contributions that are supposed to remain small for large volumes. If $\Sigma = 0$, the asymptotic behaviour of $\langle \sigma_4 \rangle_0$ dramatically changes from V^4 to V^2 , indicating a phase transition^a. On the contrary, if the quark condensate is small but non-vanishing, (11) leads, even at intermediate volumes, to a large departure from (10), and its shape is related to the detailed pattern of chiral symmetry breaking (F^2 , Σ , A , $Z_S \dots$).

These sum rules might be exploited through lattice simulations, since :

1. the N_f -dependence is explicitly displayed in the gluonic average (9),
2. renormalization group-invariant ratios of the sums σ_k can be studied,
3. computing the sums requires a set of gluonic configurations ($\nu = 0$)

and the resulting lowest Dirac e.v.s, obtained by diagonalizing the discrete version of $\mathcal{D}^2 = D^2 + \sigma_{\mu\nu} F_{\mu\nu}/2$ to avoid the problematic doublers from \mathcal{D} .

^aThis result agrees with the discussion in Sec. 2 : the $1/L^2$ -eigenvalues contribute to F^2 (SB χ S), but not to Σ . Such a situation would take place for $N_f > n_{\text{crit}}(N_c)$.

Once these sums are computed for various (large) volumes, fits with sum rules similar to (11) could determine chiral order parameters for the desired point in the (N_f, N_c) plane, until a full restoration of chiral symmetry.

4 Conclusion

Euclidean QCD is a powerful tool to investigate chiral symmetry, since its spontaneous breaking is reflected by the lowest eigenvalues of the Dirac operator. This framework provides a qualitative understanding of the N_f -dependence of infrared-dominated order parameters. In particular, the observed Zweig rule violation in the scalar channel could be explained by a large variation of the quark condensate from 2 to 3 flavours, close to a chiral phase transition. If forthcoming experiments³ related to $\pi - \pi$ scattering should soon pin down $\Sigma(2)$, an extensive study of $\text{SB}\chi\text{S}$ in the (N_f, N_c) plane remains necessary, and could be undertaken through Leutwyler-Smilga sum rules joined with lattice computations.

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